

# EQUILIBRIUM STATES FOR SMOOTH MAPS

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**ABSTRACT.** We prove an equidistribution result for  $C^\infty$  maps with respect to equilibrium states. We apply the result to the time-one map of the geodesic flow of a closed smooth Riemannian manifold.

## 1. INTRODUCTION AND MAIN RESULTS

The techniques of large deviations are widely used in the theory of dynamical systems to describe their statistical properties (see for example [6], [9] and [10]). In this paper we will use these techniques to prove an equidistribution result for smooth maps (Theorem 1 (3)) with respect to equilibrium states, i.e invariant measures which maximize the topological pressure. Furthermore, we will prove that the proportion of orbits supporting a Dirac measure close to an equilibrium state is close to 1 (Theorem 1 (2)). The general method combines the ideas in [1] for the geodesic flow, where a more geometric point of view was considered. The main result (Theorem 1) applies to the time-one map of the geodesic flow of compact Riemannian manifold and as a consequence we give a new version of the results in [1] (Corollary 1).

The process for which the large deviations are computed lies in the space of probability measures of the phase space. Let  $X$  be a compact metric space and  $\mathcal{P}(X)$  the space of probability measures on  $X$  endowed with the topology of weak convergence of measures. Let  $\{\nu_n\}_{n \in \mathbb{N}}$  be a family of probability measures on  $\mathcal{P}(X)$ . A large deviation principle for this process consists on the two following bounds: for any closed subset  $K$  and any open subset  $O$  of  $\mathcal{P}(X)$ ,

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_n(K) \leq -J(K), \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \nu_n(O) \geq -J(O)$$

where  $J$  is some positive functional defined on subsets of  $\mathcal{P}(X)$ . In what follows, we will give a precise description of (1) for smooth maps on a compact manifold  $X$  (Theorem 1 (1) and (4)). Under some conditions we will prove a contraction principle for  $\{\nu_n\}_{n \in \mathbb{N}}$  which is a large

deviation principle with constraints (Theorem 2). While a direct proof is given for the upper bound in (1), the lower bound in (1) will follow from this contraction principle. In other words, to obtain the lower bound, we have to prove a finite dimensional version of the large deviation lower bound (see Theorem 2). The main tool in this work is a formula by Kozlovski [8] for the topological pressure of a  $C^\infty$ -map (Theorem 3 below). This formula suggests that, at least when there is a unique equilibrium state, the convergence will take place exponentially fast, but it is not clear for what reasonable class of potentials. Certainly we have to impose some hyperbolic structure on the dynamics but it is remarkable that Kozlovski's formula only requires smoothness.

**1.1. Preliminaries and notations.** Let  $X$  be a smooth compact manifold and  $f : X \rightarrow X$  a  $C^\infty$  map. We assume that  $X$  has volume one,  $\int_X dx = 1$ . We denote by  $\mathcal{P}(X)$  the space of probability measures on  $X$  endowed with the weak star topology. Let  $\mathcal{P}_{inv}(X)$  be the subset of  $\mathcal{P}(X)$  of  $f$ -invariant probability measures. Given a potential  $\gamma \in C_{\mathbb{R}}(X)$ , the topological pressure of  $\gamma$  is the number defined by the variational principle [18],

$$(2) \quad P(\gamma) = \sup_{m \in \mathcal{P}_{inv}(X)} \left( h(m) + \int_X \gamma dm \right),$$

where  $h(m)$  is the entropy of the measure  $m$ . For  $F = 0$  this reduces to  $P(0) = \sup_{m \in \mathcal{P}_{inv}(X)} h(m) := h_{top}$ , where  $h_{top}$  is the topological entropy of  $f$ . An equilibrium state for  $\gamma$  is a measure  $m \in \mathcal{P}_{inv}(X)$  which achieves the maximum in (2):

$$h(m) + \int_X \gamma dm = P(\gamma).$$

We denote by  $\mathcal{P}_e(\gamma)$  the subset of  $\mathcal{P}_{inv}(X)$  of equilibrium states corresponding to  $\gamma$ . By a result of Newhouse [12], since  $f$  is  $C^\infty$ , the entropy map  $m \rightarrow h(m)$  is upper semicontinuous. Then  $h_{top} < \infty$  and consequently, the set  $\mathcal{P}_e(\gamma)$  is a nonempty closed, compact, convex subset of  $\mathcal{P}(X)$  [18].

We define the functional  $Q_\gamma$  on  $C_{\mathbb{R}}(X)$  based on the potential  $\gamma$  by,

$$(3) \quad Q_\gamma(\omega) := P(\gamma + \omega) - P(\gamma).$$

By definition,  $Q_\gamma$  is continuous on continuous functions (see Lemma 1). Sometimes we will simply write  $Q$ , if there is no confusion to be feared. We set for any probability measure  $\mu$  on  $X$ ,

$$(4) \quad J_\gamma(\mu) := \sup_{\omega} \left( \int \omega d\mu - Q_\gamma(\omega) \right),$$

where the sup is taken over the space of continuous functions  $\omega$  on  $X$ . Observe that since  $Q_\gamma(0) = 0$ , then  $J_\gamma$  is a non negative functional and clearly is lower semicontinuous. We will see that (Lemma 1) that  $\mu \in \mathcal{P}_e(\gamma)$  if and only if  $\mu$  is invariant and  $J_\gamma(\mu) = 0$ . Again, if there is no ambiguity we write  $J$  instead of  $J_\gamma$ . Since  $J$  is convex and lower semicontinuous, we have by duality,

$$(5) \quad Q_\gamma(\omega) = \sup_{\mu \in \mathcal{P}(X)} \left( \int \omega d\mu - J_\gamma(\mu) \right).$$

For any set  $E \subset \mathcal{P}(X)$  put

$$J_\gamma(E) := \inf_{\mu \in E} J_\gamma(\mu).$$

**1.2. The results.** We do the following assumption under which we prove the lower bound part of Theorem 1 (see also [6] and [4]).

*Assumption A.* *There exists a countable set  $\mathcal{C}$  of functions  $\{g_k, k \geq 1\} \subset C_{\mathbb{R}}(X)$  such that their span is dense in  $C_{\mathbb{R}}(X)$  with respect to the topology of uniform convergence,  $\|g_k\| = 1$  for all  $k$ , and for all  $\beta \in \mathbb{R}^n$  the potential  $\sum_{k=1}^n \beta_k g_k$  has a unique equilibrium state.*

This assumption holds in particular for Hölder continuous functions when the dynamical system is strongly hyperbolic. This is the case for Anosov flows (in particular for the geodesic flow of negatively curved compact manifold) and hyperbolic diffeomorphisms [17] [5].

Given  $x \in X$  we set  $\delta_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$ , where  $\delta_y$  is the Dirac measure at  $y$  and,  $\int_X \omega d\delta_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \omega(f^i(x))$ . Sometimes, to simplify the expressions we will simply denote it by  $\delta_n(x)(\omega)$ .

We also define the measures  $l_n(dx) := \|\wedge(D_x f^n)\| dx$  for each  $n \geq 1$ , where  $\|\wedge(D_x f^n)\|$  can be seen as the maximum of the volume of the images under  $D_x f^n$  of the arbitrary dimensional cubes with volume 1. More formally,  $\|\wedge(D_x f^n)\| := \max_{j \leq \dim X} \|\wedge^j(D_x f^n)\|$ , with

$$\|\wedge^j(D_x f^n)\| = \max_{V \in Gr(\dim X, j)} |\det(D_x f^n|_V)|,$$

where  $Gr(\dim X, j)$  (the grassmannian) is the set of all the subspace of  $T_x X$  of dimension  $j$ .

**Theorem 1.** *Let  $f : X \rightarrow X$  be a  $C^\infty$  map of a smooth compact manifold  $X$ . Then, for any continuous function  $\gamma \in C_{\mathbb{R}}(X)$  we have:*

(1) *For any closed subset  $K$  of  $\mathcal{P}(X)$*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in K)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \leq -J(K).$$

(2) For any open neighborhood  $V$  of  $\mathcal{P}_e(\gamma)$  we have,

$$\lim_{n \rightarrow \infty} \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in V)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} = 1,$$

where the convergence is exponential with speed  $e^{-nJ(V^c)}$ .

(3) The weak limits of

$$\mu_n := \frac{\int_X e^{n \int \gamma d\delta_n(x)} \delta_n(x) l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}$$

are equilibrium states corresponding to the potential  $\gamma$ , i.e any weak limit  $\mu_\infty$  is an invariant probability measure and satisfies

$$h(\mu_\infty) + \int_X \gamma d\mu_\infty = P(\gamma).$$

(4) Assume A. If for all  $\beta \in \mathbb{R}^d$  and  $g = (g_1, \dots, g_d) \in \mathcal{C}^d$ ,  $\gamma + \beta \cdot g$  has a unique equilibrium state, then for any open subset  $O$  of  $\mathcal{P}(X)$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in O)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \geq -J(O).$$

The upper bound (1) can be interpreted as a measurement of the growth of the  $l_n$ -volume of certain sets as follows. Let us write it for  $\gamma = 0$  for simplicity: for  $n$  sufficiently large we get from (1),

$$e^{-nl_n(X)} l_n(\{x \in X : \delta_n(x) \in K\}) \leq e^{-nJ_0(K)}$$

where  $J_0$  is the functional  $J_\gamma$  corresponding to  $\gamma = 0$ . Recall from (4) that in this case we have for any probability measure  $m$  on  $X$ ,  $J_0(m) = \sup_\omega (\int_X \omega dm - P(\omega)) + h_{top}$ . Furthermore, if  $K$  contains an invariant probability measure  $\mu_K$  and  $J_0$  achieves its minimum in  $K$  at  $\mu$ ,  $J_0(K) = J_0(\mu_K)$ , then by Lemma 1 hereunder we will have  $J_0(\mu) = h_{top} - h(\mu_K)$ . Thus for  $n$  sufficiently large we get,

$$e^{-nl_n(X)} l_n(\{x \in X : \delta_n(x) \in K\}) \leq e^{-n(h_{top} - h(\mu_K))}.$$

Point (3) of this theorem will be deduced from the point (1) and there is no need of *Assumption A*.

Theorem 1 apply in particular to the time-one map of the geodesic flow of the unit tangent bundle of a compact Riemannian manifold  $M$ . In this case, the weak limits of  $\mu_n$  are equilibrium states of the geodesic flow corresponding to the potential  $\gamma$ . If  $M$  is a manifold of negative curvature, it is well known that for any Hölder potential there exists a unique equilibrium state, then the corresponding measures  $\mu_n$  converge to this state. There are three well known invariant measures in this

setting. The Bowen-Margulis measure  $\mu_0$ , which is the equilibrium state (a measure of maximal entropy) corresponding to the constant potential. The harmonic measure  $\nu$  which corresponds to the potential  $\frac{d}{dt}|_{t=0}(K \circ \tilde{\varphi}_t)$  where  $K$  is the Poisson kernel and  $\tilde{\varphi}_t$  is the geodesic flow of  $\widetilde{SM}$ . The Liouville measure  $m_{liou}$  which is the equilibrium state of the potential  $\frac{d}{dt}|_{t=0} \det(d\varphi_t|_{E^s})$  where  $E^s$  is the stable tangent bundle of  $SM$  (see [2] and [3] for more details). If  $M$  is a rank 1 manifold (Riemannian manifolds of nonpositive curvature), Knieper [7] showed that there exists a uniquely determined invariant measure of maximal entropy  $\mu_{\max}$  for the geodesic flow and then  $\mu_n$  converges towards  $\mu_{\max}$ . As a consequence of Theorem 1 we deduce the following result for the geodesic flow.

**Corollary 1.** *Let  $M$  be a closed and connected manifold equipped with a  $C^\infty$  Riemannian metric and  $f \equiv \varphi_1$  the time-one map of the geodesic flow  $\phi$  of the unit tangent bundle  $X = SM$  of  $M$ . Let  $\gamma \in C_{\mathbb{R}}(SM)$  be a continuous potential. Then the weak limits of*

$$\mu_n := \frac{\int_{SM} e^{n \int \gamma d\delta_n(\theta)} \delta_n(\theta) l_n(d\theta)}{\int_{SM} e^{n \int \gamma d\delta_n(\theta)} l_n(d\theta)}$$

*are equilibrium states for the geodesic flow corresponding to the potential  $\gamma$ . Here  $d\theta$  is the volume form induced by the Riemannian metric on the tangent bundle  $TM$  and  $l_n(d\theta) := \|\wedge(D_\theta \varphi_1^n)\| d\theta$ .*

**1.3. Contraction Principle.** For  $g \in C_{\mathbb{R}^d}(X)$  and  $\alpha \in \mathbb{R}^d$  we set  $m(g) := \int g dm$  and

$$\mathcal{P}_{g,\alpha}(X) := \{m \in \mathcal{P}(X) : m(g) = \alpha\}.$$

We define the functionals:

$$J_g(\alpha) = \begin{cases} \inf(J(m) : m \in \mathcal{P}_{g,\alpha}(X)) & \text{if } \mathcal{P}_{g,\alpha}(X) \neq \emptyset \\ +\infty & \text{if } \mathcal{P}_{g,\alpha}(X) = \emptyset \end{cases}$$

and  $J_g(E_d) = \inf(J_g(\alpha) : \alpha \in E_d)$  for any  $E_d \subset \mathbb{R}^d$ . The following result is known as a contraction principle [4].

**Theorem 2** (Contraction Principle). *Let  $f : X \rightarrow X$  a  $C^\infty$  map of a smooth compact manifold  $X$ . Let  $\gamma : X \rightarrow \mathbb{R}$  be a continuous potential and  $g \in C_{\mathbb{R}^d}(X)$ . Then,*

(1) *For any closed subset  $K_d \subset \mathbb{R}^d$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x)(g) \in K_d)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \leq -J_g(K_d).$$

- (2) If for all  $\beta \in \mathbb{R}^d$ ,  $\gamma + \beta \cdot g$  has a unique equilibrium state, then for any open subset  $O_d \subset \mathbb{R}^d$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_d)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \geq -J_g(O_d).$$

## 2. PROOFS

**2.1. Preliminaries.** For any invariant probability measure  $\mu$  we set

$$(6) \quad I(\mu) := P(\gamma) - (h(\mu) + \int_{SM} \gamma d\mu).$$

**Lemma 1.** (1)  $Q_\gamma$  is  $f$ -invariant, that is  $Q_\gamma(\omega \circ f) = Q_\gamma(\omega)$  for all continuous function  $\omega$ . Moreover,  $Q_\gamma$  is convex and continuous on continuous functions.

- (2)  $Q_\gamma(\omega) = \sup_{\mu \in \mathcal{P}_{inv}(SM)} (\int \omega d\mu - I(\mu))$ . In other words, the functionals  $I$  and  $J_\gamma$  agree on invariant measures.

- (3)  $\mathcal{P}_e(\gamma) = \{J_\gamma = 0\}$ , that is  $J_\gamma$  vanishes on a probability measure  $\mu$  if and only if  $\mu$  is an equilibrium state for  $f$ .

*Proof.* The fact that  $Q_\gamma$  is  $f$ -invariant follows from its definition (3). Part (2) is a consequence of the convexity of the pressure function  $P$  and the variational principle (1) from which we can easily deduce that  $|P(f) - P(g)| \leq \|f - g\|_\infty$  [18]. Part (2) follows from (7) and,

$$\begin{aligned} & \sup_{\mu \in \mathcal{P}_{inv}(X)} \left( \int \omega d\mu - I(\mu) \right) \\ &= \sup_{\mu \in \mathcal{P}_{inv}(X)} \left( \int \omega d\mu - P(\gamma) + h(\mu) + \int \gamma d\mu \right) \\ &= P(\gamma + \omega) - P(\gamma) = Q_\gamma(\omega). \end{aligned}$$

Thus, a probability measure  $m$  satisfies  $J_\gamma(m) = 0$  if and only if  $m$  is invariant and  $I(m) = 0$ . On the other hand, by definition and the continuity of  $Q_\gamma$ , the functional  $J_\gamma$  is lower semicontinuous. Thus, if  $K$  is a closed subset of  $\mathcal{P}(X)$  we have  $\inf_{m \in K} J_\gamma(m) = J_\gamma(\mu)$  for some  $\mu \in K$ . Then  $\inf_{m \in K} J_\gamma(m) = 0$  iff  $\mu \in K$  is invariant and  $h(\mu) + \int \gamma d\mu = P(\gamma)$ . In other words, by Lemma 1 (2), we have  $\mathcal{P}_e(\gamma) = \{J_\gamma = 0\}$ .  $\square$

The proofs below are based on a formula of Kozlovski [8] which asserts that the topological pressure for a  $\mathcal{C}^\infty$  map of a smooth compact manifold is given by the exponential growth of the mapping  $\wedge(D_x f^n)$  between full exterior algebras of the tangent spaces.

**Theorem 3** (O. S. Kozlovski). *Let  $f : X \rightarrow X$  be a  $C^\infty$  map of a smooth compact manifold  $X$ . Then, the topological pressure  $P(\gamma)$  of a potential  $\gamma \in C_{\mathbb{R}}(X)$  is given by*

$$(7) \quad P(\gamma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_X e^{n \int \gamma d\delta_n(x)} l_n(dx).$$

## 2.2. Proof of Theorem 1 (1).

*Proof.* Set for any subset  $E$  of  $\mathcal{P}(X)$ ,

$$(8) \quad \nu_n(E) := \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in E)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}.$$

We have to prove

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_n(K) \leq - \inf_{m \in K} J(m) := -J(K)$$

for any closed subset  $K$ . Let  $\epsilon > 0$ . Observe that the set  $K$  is contained the union of open sets,

$$K \subset \cup_{\omega} \{ \mu \in \mathcal{P}(X) : \int \omega d\mu - Q(\omega) > J(K) - \epsilon \}.$$

There exists then a finite number of continuous functions  $\omega_1, \dots, \omega_l$  such that  $K \subset \cup_{i=1}^l K_i$ , where

$$K_i = \{ m \in \mathcal{P}(X) : \int \omega_i dm - Q(\omega_i) > J(K) - \epsilon \}.$$

Put

$$\Gamma_i(x, n) := \{ x \in X : \delta_n(x) \in K_i \}, \text{ and } Z_i(n) := \int_{\Gamma_i(x, n)} e^{n \int \gamma d\delta_n(x)} l_n(dx)$$

We have  $\nu_n(K) \leq \sum_{i=1}^l \nu_n(K_i)$ , where  $\nu_n(K_i) = \frac{Z_i(n)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}$ . By definition of  $Q$  and  $\Gamma_i(x, n)$ , and Theorem 3 we have for  $n$  sufficiently

large,

$$\begin{aligned}
& \nu_n(K) \\
& \leq \sum_{i=1}^l Z_i(n) e^{-n(P(\gamma)-\epsilon)} \\
& \leq e^{-n(P(\gamma)-\epsilon)} \sum_{i=1}^l \int_{\Gamma_i(x,n)} e^{n \int \gamma d\delta_n(x)} e^{n(\int \omega_i d\delta_n(x) - Q(\omega_i) - (J(K)-\epsilon))} l_n(dx) \\
& \leq e^{n(-Q(\omega_i) - (J(K)-\epsilon))} e^{-n(P(\gamma)-\epsilon)} \sum_{i=1}^l \int_{\Gamma_i(x,n)} e^{n \int (\gamma + \omega_i) d\delta_n(x)} l_n(dx) \\
& \leq \sum_{i=1}^l e^{-n(P(\gamma)-\epsilon)} e^{n(-Q(\omega_i) - (J(K)-\epsilon))} e^{n(P(\gamma + \omega_i) + \epsilon)} \\
& = l e^{n(-J(K) + 3\epsilon)}.
\end{aligned}$$

Take the logarithme, divide by  $n$  and the lim sup,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_n(K) \leq -J(K) + 3\epsilon.$$

$\epsilon$  being arbitrary, this proves Theorem 1 (1).  $\square$

### 2.3. Proof of Theorem 1 (2).

*Proof.* It is a consequence of Theorem 1 (1). Indeed, we have to prove that  $\lim_{n \rightarrow \infty} \nu_n(V) = 1$ . Set  $K = V^c$  the complement of  $V$  in  $\mathcal{P}(X)$ . We have  $J(K) = J(m)$  for some  $m \in K$ . Thus by Lemma 1,  $J(K) > 0$  and for  $n$  sufficiently large,

$$1 \geq \nu_n(V) \geq 1 - e^{-nJ(K)}.$$

$\square$

### 2.4. Proof of Theorem 1 (3).

*Proof.* We have to show that the weak limits of

$$\mu_n := \frac{\int_X e^{n \int \gamma d\delta_n(x)} \delta_n(x) l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}$$

are contained in  $\mathcal{P}_e(\gamma)$ . Observe that we can write  $\mu_n = E_{\beta_n}(\delta_n)$  where the expectation is taken with respect to the probability measure on  $X$ ,

$$\beta_n(E) := \frac{\int_E e^{n \int \gamma d\delta_n(x)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}.$$

This means that we view  $\delta_n$  as a random variable on the probability space  $(X, \beta_n)$ .



Let  $V \subset \mathcal{P}(X)$  be a convex open neighborhood of  $\mathcal{P}_e(\gamma)$  and  $\epsilon > 0$ . We consider a finite open cover  $(B_i(\epsilon))_{i \leq N}$  of  $\mathcal{P}_e(\gamma)$  by balls of diameter  $\epsilon$  all contained in  $V$ .

Decompose the set  $U := \cup_{i=1}^N B_i(\epsilon)$  as follows,

$$U = \cup_{j=1}^{N'} U_j^\epsilon, \quad N' \geq N,$$

where the sets  $U_j^\epsilon$  are disjoint and contained in one of the balls  $(B_i(\epsilon))_{i \leq N}$ . We have

$$\mathcal{P}_e(\gamma) \subset U \subset V,$$

and  $\sum_{j=1}^{N'} \nu_n(U_j^\epsilon) = \nu_n(U)$ . We fix in each  $U_j^\epsilon$  a probability measure  $p_j$ ,  $j \leq N'$ , and let  $p_0$  be a probability measure distinct from the above ones (for example take  $p_0 \in V \setminus U$ ).

Define the following process on the space  $(X, \beta_n)$ ,

$$\omega_n := \sum_{j=1}^{N'} p_j 1_{\delta_n^{-1}(U_j^\epsilon)} + (1 - \nu_n(U)) p_0.$$

We have,

$$(9) \quad E_{\beta_n}(\omega_n) = \sum_{j=1}^{N'} \nu_n(U_j^\epsilon) p_j + (1 - \nu_n(U)) p_0.$$

Since  $\text{span}\{g_1, g_2, \dots\}$  is dense in  $C_{\mathbb{R}}(X)$ , the topology generated by the metric:

$$d(m, m') := \sum_{k=1}^{\infty} 2^{-k} \left| \int g_k dm - \int g_k dm' \right|,$$

is compatible with weak star topology. We will use it to evaluate the distance between elements in  $\mathcal{P}(X)$ .

The probability measure  $E_{\beta_n}(\omega_n)$  lies in  $V$  since it is a convex combination of elements of the convex set  $V$ . We have then  $d(\mu_n, V) \leq d(\mu_n, E_{\beta_n}(\omega_n))$ . We will show that

$$d(\mu_n, E_{\beta_n}(\omega_n)) \leq \epsilon \nu_n(U) + \frac{3}{2} \nu_n(U^c),$$

where  $U^c = \mathcal{P}(X) \setminus U$  which is closed.

Set  $\mu_{n,V} := E_{\beta_n}((1_V \circ \delta_n) \delta_n)$  (this defines a finite measure on  $X$ ). By definition of  $\mu_n$  and  $\mu_{n,V}$  and the fact that  $U \subset V$ , we get

$$\sum_{k \geq 1} 2^{-k} |\mu_n(g_k) - \mu_{n,V}(g_k)| \leq \frac{1}{2} \nu_n(U^c).$$

It remains to show that

$$\sum_{k \geq 1} 2^{-k} |\mu_{n,V}(g_k) - E_{\beta_n}(\omega_n)(g_k)| \leq \epsilon \nu_n(U) + \nu_n(U^c).$$

We have for all  $k \geq 1$ ,

$$|\mu_{n,V}(g_k) - E_{\beta_n}(\omega_n)(g_k)| \leq A + B + C$$

where,

$$\begin{aligned} A &= \frac{\sum_{j=1}^{N'} \int_{(x:\delta_n(x) \in U_j^\epsilon)} e^{n \int \gamma d\delta_n(x)} |\delta_n(x)(g_k) - m_j(g_k)| l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}, \\ B &= \frac{\int_{(x:\delta_n(x) \in V \setminus U)} e^{n \int \gamma d\delta_n(x)} \delta_n(x)(g_k) l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \\ C &= |(1 - \nu_n(U))m_0(g_k)|. \end{aligned}$$

Thus, since we have for all  $k \geq 1$ ,  $\|g_k\| = 1$ , by definition of  $\nu_n$  we get,

$$\begin{aligned} &\sum_{k \geq 1} 2^{-k} |\mu_{n,V}(g_k) - E_{\beta_n}(\omega_n)(g_k)| \\ &\leq \epsilon \sum_{j=1}^{N'} \nu_n(U_j^\epsilon) + \frac{1}{2} \nu_n(U^c) + \frac{1}{2} (1 - \nu_n(U)) \\ &= \epsilon \nu_n(U) + \nu_n(U^c). \end{aligned}$$

Finally we have obtained that

$$\begin{aligned} d(\mu_n, E_{\beta_n}(\omega_n)) &= \sum_{k \geq 1} 2^{-k} |\mu_n(g_k) - E_{\beta_n}(\omega_n)(g_k)| \\ &\leq \epsilon \nu_n(U) + \frac{3}{2} \nu_n(U^c). \end{aligned}$$

This implies the desired inequality,

$$d(\mu_n, V) \leq \epsilon \nu_n(U) + \frac{3}{2} \nu_n(U^c).$$

By Theorem 1 (2), since  $U^c$  is closed, we know that  $\lim_{n \rightarrow \infty} \nu_n(U) = 1$ . Thus,  $\limsup_{n \rightarrow \infty} d(\mu_n, V) \leq \epsilon$ , for all  $\epsilon > 0$ . We conclude that  $\limsup_{n \rightarrow \infty} d(\mu_n, V) = 0$ . The neighborhood  $V$  of  $\mathcal{P}_e(\gamma)$  being arbitrary, this implies that all limit measures of  $\mu_n$  are contained in  $\mathcal{P}_e(\gamma)$ . In particular, if  $\mathcal{P}_e(\gamma)$  is reduced to one measure  $\mu$ , this shows that  $\mu_n$  converges to  $\mu$ .  $\square$

## 2.5. Proof of Theorem 2.

2.5.1. *Proof of part (1).*

*Proof.* The map  $g \rightarrow \delta_n(x)(g)$  being continuous, Theorem 2 (1) follows from Theorem 1 (1).  $\square$

2.5.2. *Proof of part (2).*

*Proof.* If  $J_g(O_d) = +\infty$  then there is nothing to do. Suppose then  $J_g(O_d) < +\infty$ . Let  $\varepsilon > 0$  and choose  $\alpha_\varepsilon \in O_d$  with  $\mathcal{P}_{g,\alpha_\varepsilon}(X) \neq \emptyset$  such that

$$J_g(O_d) > I_g(\alpha_\varepsilon) - \varepsilon,$$

We know from ([16] Theorem 23.4 and 23.5) that, given  $\alpha$  in the interior of the affine hull of the domain  $D(J_g)$  of  $J_g$ , there exists  $\beta \in \mathbb{R}^d$  such that

$$Q_\gamma(\beta \cdot g) = \beta \cdot \alpha - J_g(\alpha).$$

Let then  $\beta_\varepsilon \in \mathbb{R}^d$  such that

$$(10) \quad Q_\gamma(\beta_\varepsilon \cdot g) = \beta_\varepsilon \cdot \alpha_\varepsilon - J_g(\alpha_\varepsilon).$$

Consider now a small neighborhood of  $\alpha_\varepsilon$ ,

$$O_{d,r} := \{\alpha \in \mathbb{R}^d : |\alpha_\varepsilon - \alpha| \leq r\},$$

such that  $O_{d,r} \subset O_d$ . Set

$$\psi_n(O_d) := \int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_d)} l_n(dx),$$

and  $Z_n(O_d) := \frac{\psi_n(O_d)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}$ . We have,  $Z_n(O_d) \geq Z_n(O_{d,r})$  and

$$\psi_n(O_{d,r}) \geq e^{-n\beta_\varepsilon \cdot \alpha_\varepsilon} e^{-n\|\beta_\varepsilon\|r} \int_X e^{n \int (\gamma + \beta_\varepsilon \cdot g) d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_{d,r})} l_n(dx)$$

Thus

$$Z_n(O_{d,r}) \geq e^{-n\beta_\varepsilon \cdot \alpha_\varepsilon} e^{-n\|\beta_\varepsilon\|r} \frac{\int_X e^{n \int (\gamma + \beta_\varepsilon \cdot g) d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_{d,r})} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}.$$

Set

$$Z_n^\varepsilon(O_{d,r}) := \frac{1}{n} \log \frac{\int_X e^{n \int (\gamma + \beta_\varepsilon \cdot g) d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_{d,r})} l_n(dx)}{\int_X e^{n \int (\gamma + \beta_\varepsilon \cdot g) d\delta_n(x)} l_n(dx)}$$

and

$$Z_n(\beta_\varepsilon \cdot g) := \frac{1}{n} \log \frac{\int_X e^{n \int (\gamma + \beta_\varepsilon \cdot g) d\delta_n(x)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)}.$$

Therefore,

$$\frac{1}{n} \log Z_n(O_{d,r}) \geq -r\|\beta_\varepsilon\| + (Z_n(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha_\varepsilon) + \frac{1}{n} \log Z_n^\varepsilon(O_{d,r}).$$

From Kozlovski's theorem we deduce that

$$\lim_{n \rightarrow \infty} Z_n(\beta_\varepsilon \cdot g) = P(\gamma + \beta_\varepsilon \cdot g) - P(\gamma) = Q_\gamma(\beta_\varepsilon \cdot g).$$

Thus

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log Z_n(O_{d,r}) \geq -r \|\beta_\varepsilon\| + (Q_\gamma(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha_\varepsilon) + \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^\varepsilon(O_{d,r}).$$

We will show that

$$(11) \quad \lim_{n \rightarrow \infty} Z_n^\varepsilon(O_{d,r}) = 1.$$

Let us show how to finish the proof using (11) :

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log Z_n(O_d) &\geq -r \|\beta_\varepsilon\| + Q_\gamma(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha_\varepsilon \\ &= -r \|\beta_\varepsilon\| - J_g(\alpha_\varepsilon) \\ &\geq -r \|\beta_\varepsilon\| - J_g(O_d) - \varepsilon, \end{aligned}$$

for any  $\varepsilon > 0$ . Since  $r > 0$  was arbitray choosen, we let  $r \rightarrow 0$  and  $\varepsilon \rightarrow 0$  respectively and we get  $\liminf_{T \rightarrow \infty} \frac{1}{T} \log Z_T(O_d) \geq -J_g(O_d)$  which completes the proof of Theorem 2 (2).

It remains to show (11). Let  $K_{d,r}$  be the complement set of  $O_{d,r}$  in the image  $g * (\mathcal{P}_{inv}(X))$  of  $\mathcal{P}_{inv}(X)$  under the continuous map  $g * : m \rightarrow m(g)$ . We have  $Z_n^\varepsilon(O_{d,r}) + Z_n^\varepsilon(K_{d,r}) = 1$ . The goal is to show using Theorem 1 that,  $Z_n^\varepsilon(K_{d,r})$  decrease to zero as  $n \rightarrow \infty$  (in fact exponentially fast).

Consider  $J^\varepsilon := J_{\gamma + \beta_\varepsilon \cdot g}$ , which is the functional  $J$  corresponding to  $Q^\varepsilon := Q_{\gamma + \beta_\varepsilon \cdot g}$ . By definitions (3) and (4), we have for any continuous function  $\omega$  on  $X$ ,

$$Q^\varepsilon(\omega) = P(\gamma + \beta_\varepsilon \cdot g + \omega) - P(\gamma + \beta_\varepsilon \cdot g),$$

and for any probability measure  $m$  on  $X$ ,

$$J^\varepsilon(m) = \sup_{\omega} \left( \int \omega dm - Q^\varepsilon(\omega) \right).$$

From this we deduce easily that

$$J^\varepsilon(m) := J_\gamma(m) + Q_\gamma(\beta_\varepsilon \cdot g) - \int \beta_\varepsilon \cdot g dm,$$

and

$$\inf_{m(g)=\alpha} J^\varepsilon(m) = \inf_{m(g)=\alpha} J_\gamma(m) + Q_\gamma(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha.$$

The set  $K_{d,r}$  is compact in  $\mathbb{R}^d$ , then by Theorem 1 (1) we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n^\varepsilon(K_{d,r}) \leq -J^\varepsilon(K),$$

where  $K := (g^*)^{-1}(K_{d,r})$  which is a closed subset of  $\mathcal{P}(X)$ . If  $J^\varepsilon(K) = +\infty$  there is nothing to do and the result follows. The key point is to prove that  $J^\varepsilon(K) > 0$ . For this, set

$$J_g^\varepsilon(\alpha) := J_g(\alpha) + Q(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha.$$

The functional  $J^\varepsilon$  is non negative (since  $Q^\varepsilon(0) = 0$ ), lower semicontinuous and then it achieves its minimum on compact sets. We have  $J_g^\varepsilon(\alpha) \geq 0$  and  $J_g^\varepsilon(\alpha_\varepsilon) = 0$  (see (10)). Recall that, if  $J_g^\varepsilon(\alpha) = 0$  for some  $\alpha \in K_{d,r}$ , then there will correspond to  $\alpha$  an equilibrium state  $m_\alpha \in K$  for the potential  $\gamma + \beta_\varepsilon \cdot g$  such that  $m_\alpha(g) = \alpha$ . The vector  $\alpha_\varepsilon$  is the unique point realizing the minimum, i.e the unique solution for the equation  $J_g^\varepsilon(\alpha) = 0$ . Indeed, two different solutions will produce two distinct equilibrium states for the potential  $\gamma + \beta_\varepsilon \cdot g$  which contradicts our standing assumption of Theorem 1. Since  $\alpha_\varepsilon \in O_{d,r}$ , then  $J_g^\varepsilon(\alpha) > 0$  for  $\alpha \in K_{d,r}$ . On the other hand the set  $K_{d,r}$  being compact, by the lower semicontinuity of  $J^\varepsilon$  we have  $J^\varepsilon(K) = \inf_{\alpha \in K_{d,r}} J_g^\varepsilon(\alpha) > 0$ . Thus we have proved that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n^\varepsilon(K_{d,r}) \leq -J^\varepsilon(K) < 0$$

from which (11) follows immediately.  $\square$

## 2.6. Proof of Theorem 1 (4).

*Proof.* Let  $O \subset \mathcal{P}(X)$  be an open set,  $\epsilon > 0$  and choose  $m_\epsilon \in O$  such that

$$I(m_\epsilon) \leq \rho(O) + \epsilon.$$

For each  $N$  we set,

$$d_N(m, m') := \sum_{k=1}^N 2^{-k} \left| \int g_k dm - \int g_k dm' \right|.$$

Set  $2r = \inf\{d(m, m_\epsilon) : m \in \mathcal{P}(X) \setminus O\}$ . We have  $r > 0$ , since  $\mathcal{P}(X) \setminus O$  is a compact subset of  $\mathcal{P}(X)$ . Since for all  $k$ ,  $\|g_k\| = 1$ , we have  $0 \leq d(m, m') - d_N(m, m') \leq 2^{-(N-1)}$ . Thus, for  $N$  sufficiently large,

$$O_{\epsilon,r} := \{m \in \mathcal{P}(X) : d_N(m, m_\epsilon) < r\} \subset O.$$

For each  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$  denote  $\|\alpha\|_N = \sum_{k=1}^N 2^{-k} |\alpha_k|$ . Let  $g = (g_1, \dots, g_N)$  and set  $\alpha_\epsilon = m_\epsilon(g) := (\int g_1 dm_\epsilon, \dots, \int g_N dm_\epsilon)$  and

$$O_{N,r} := \{\alpha \in \mathbb{R}^N : \|\alpha_\epsilon - \alpha\|_N < r\}.$$

Then,  $g(O_{\epsilon,r}) = O_{N,r} \cap g(\mathcal{P}(X))$ . From Theorem 2 (2) we get,

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in O)} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \\
& \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x) \in O_{\epsilon,r})} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \\
& = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{\int_X e^{n \int \gamma d\delta_n(x)} 1_{(\delta_n(x)(g) \in O_{N,r})} l_n(dx)}{\int_X e^{n \int \gamma d\delta_n(x)} l_n(dx)} \\
& \geq -J_g(O_{N,r}) \\
& \geq -J_g(\alpha_\epsilon) \geq -J(m_\epsilon) \geq -J(O) - \epsilon,
\end{aligned}$$

for any  $\epsilon > 0$ . This complete the proof of Theorem 1 (4).  $\square$

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